

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. State and prove the Hahn - Banach theorem.
 17. If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in H , then prove that $x - \sum (x, e_i)e_i \perp e_g$ for each j .
 18. Let H be a Hilbert space and let f be an arbitrary functional in H^* . Then prove that there exists a unique vector y in H such that, $f(x) = (x, y)$ for every x in H .
 19. Prove that $\sigma(x)$ is non-empty.
 20. State and prove the Gelfand Neumark theorem.
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NOVEMBER/DECEMBER 2025

GMA43/DMA43 — FUNCTIONAL
ANALYSIS

Time : Three hours

Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. What is normed linear space?
2. Define continuous linear functionals.
3. State uniform boundedness theorem.
4. Define orthogonal complement.
5. What is normal operators?
6. When will you say that a projection is a perpendicular projection?
7. What is Banach algebra?
8. Define Radical R .
9. What is multiplication functional?
10. Define Self-adjoint.



SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL questions.

11. (a) Let M be a closed linear subspace of a normed linear space N . If the norm of a coset $x + M$ in the quotient space N/M is defined by $\|x + M\| = \inf \{\|x + m\| : m \in M\}$ then prove that N/M is a normed linear space. Further, if N is a Banach space then so is N/M .

Or

- (b) Let M be a linear subspace of a normed linear space N and let f be a functional defined on M . If x_0 is a vector not in M , and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0 then f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

12. (a) State and prove the closed graph theorem.

Or

- (b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M + N$ is also closed.

13. (a) If T is an operator on H for which $(T_x, x) = 0$ for all x , then prove that $T = 0$.

Or

- (b) A closed linear subspace M of H is invariant under an operator $T \Leftrightarrow M^\perp$ is invariant under T^* .

14. (a) Prove that the boundary of S is a subset of Z .

Or

- (b) If r is an element of $\mathbb{R}$, then prove that $1 - r$ is regular.

15. (a) If F_1 and F_2 are multiplicative functionals on A with the same null space M , then prove that F_1 and F_2 .

Or

- (b) Prove that the following conditions on A are all equivalent to one another :

(i) $\|x^2\| = \|x\|^2$ for every x

(ii) $r(x) = \|x\|$ for every x

(iii) $\|x^{-1}\| = \|x\|$ for every x .